Partial Differential Equations Midterm Exam

March 6, 2018

You have 2 hours to complete this exam. Please show all work. Each question is worth 25 points for a total of 100 points. Be sure to quote clearly any theorems you use from the textbook or class. Good luck!

1. Solve the wave equation $\partial_t^2\phi = \partial_x^2\phi$ using D'Alembert's principle with the initial conditions:

$$a)\ \phi(x,0) = x \qquad \partial_t\phi(x,0) = x^2$$

$$b)\ \phi(x,0) = \begin{cases} 0 & \text{if } |x| > 1 \\ 1 & \text{if } |x| \le 1 \end{cases} \qquad \partial_t\phi(x,0) = \begin{cases} 0 & \text{if } |x| > 1 \\ -1 & \text{if } |x| \le 1 \end{cases}$$

2. Consider the parabolic differential equation $\partial_{xx}F + 2\partial_{xy}F + \partial_{yy}F = 0$. Let

$$u(x,y) = Px + Qy \quad v(x,y) = Rx + Sy \quad PS - QR \neq 0$$

be a linear change of coordinates. Find a choice of P, Q, R, S such that

$$\partial_{uu}F = \partial_{xx}F + 2\partial_{xy}F + \partial_{yy}F = 0$$

and hence find the general solution of the given parabolic equation in terms of x and y.

3. Consider the diffusion equation $\partial_t u = \partial_{xx}u$ in the set

$$\{(x,t) :\ 0 \le x \le 1, \quad 0 \le t < \infty\}$$

with $u(0,t) = u(1,t) = 0$ and $u(x,0) = 4x(1-x)$
a) Show that $0 < u(x,t) < 1$ for all $t > 0$ and $0 < x < 1$.
b) Show that $u(x,t) = u(1-x,t)$ for all $t \ge 0$ and $0 \le x \le 1$
c) Use the energy method to show that

$$\int_0^1 u^2(t,x)\,dx$$

is a strictly decreasing function of t.

4. The Schrodinger equation for the complex probability amplitude $\psi(x,t)$ of a free particle is

$$-ih\frac{\partial \psi}{\partial t} = -\frac{h^2}{2m}\frac{\partial^2 \psi}{\partial x^2} \tag{0.1}$$

where h, m are constants and $i = \sqrt{-1}$.
a) Separate the variables $\psi(x,t) = X(x)T(t)$ and show that $T(t) = \exp(iEt/h)$ for some constant E called the energy. (This is the separation constant in the language of partial differential equations)
b) Assume that $X(0) = X(L) = 0$ for the interval $[0, L]$. Show that the energy of the separated solution is quantised, that is, it can only take on the values

$$\frac{n^2 h^2 \pi^2}{2mL^2} \quad n \in \mathbb{Z}$$